# On a conjecture involving cycle-complete graph Ramsey numbers

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**Abstract**

It has been conjectured that $r(C_n, K_m) = (m-1)(n-1)+1$ for all $(n, m) \neq (3,3)$ satisfying $n \geq m$. We prove this for the case $m = 5$.


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# 1 Introduction

The *independence number* $\alpha(G)$ of a graph $G$ is the cardinality of its largest independent set. Given a graph $H$ without isolated vertices, the *Ramsey number* $r(H, K_m)$ is the smallest integer $N$ such that every graph $G$ of order $N$ either contains $H$ as a subgraph or satisfies $\alpha(G) \geq m$. In one of the earliest contributions to graphical Ramsey theory [1], Bondy and Erdős proved the following result for the case where $H \cong C_n$, a cycle of length $n$.

**Theorem (Bondy, Erdős).** *For all* $n \geq m^2 - 2$,

$$r(C_n, K_m) = (m-1)(n-1) + 1.$$

The condition $n \geq m^2 - 2$ is required because of the proof technique, and it has been thought from the beginning that the conclusion is likely to hold under a rather less restrictive hypothesis. The problem of determining for each $m$ the smallest $n$ for which $r(C_n, K_m) = (m-1)(n-1) + 1$ is among those given in [3], and it is conjectured in [8] and elsewhere that $r(C_n, K_m) = (m-1)(n-1) + 1$ for all $(n, m) \neq (3, 3)$ satisfying $n \geq m$. This is trivial for $m = 2$. It was confirmed for $m = 3$ in early work on graphical Ramsey theory [4], and recently it was proved for $m = 4$ [9]. In this paper, we shall prove that the conjecture is true for $m = 5$.

**Theorem 1.** *For all* $n \geq 5$, $r(C_n, K_5) = 4n - 3$.

*Note.* The condition $n \geq 5$ is best possible. From early work of Clancy [2], it is known that $r(C_4, K_5) = 14$. There is a unique graph $G$ of order 13 such that $C_4 \not\subset G$ and $\alpha(G) \leq 4$. This graph is exhibited in [6] and elsewhere.

To reach our goal, it is only necessary to prove that for $n \geq 5$ every $C_n$-free graph $G$ of order $4n - 3$ satisfies $\alpha(G) \geq 5$. The fact that $r(C_n, K_5) \geq 4n - 3$ follows from the simple example of $G \cong 4K_{n-1}$, which contains no $C_n$ and has independence number $\alpha(G) = 4$.

# 2 Proofs

The proof of Theorem 1 will be given through a sequence of lemmas. As usual, $\delta(G)$ denotes the minimum degree, that is $\delta(G) = \min_{v \in V(G)} \deg v$.

**Lemma 1.** *Suppose that for some* $n \geq 4$ *there exists a graph* $G$ *of order* $4(n-1)+1$ *such that* $C_n \not\subset G$ *and* $\alpha(G) \leq 4$. *Then* $\delta(G) \geq n - 1$.

*Proof.* Suppose to the contrary that some vertex $v \in V(G)$ satisfies $\deg v \leq n - 2$. Deleting $v$ and its neighborhood, we obtain a graph $H$ of order at least $3(n-1)+1$. By the result in [9] either $C_n \subset H$ or $\alpha(H) \geq 4$. Since $C_n \not\subset G$, we must assume that latter. But then $v$ together with the appropriate four vertices from $V(H)$ yields a five-element independent set in $G$, a contradiction. $\square$

The following lemma is proved in [7].

**Lemma 2.** *Suppose $\delta(G) \geq 4$ and $C_5 \not\subset G$. Then $\alpha(G) \geq \Delta(G)$.*

The following result is given in [5]. In the interest of completeness, it is included here with proof.

**Lemma 3.** $r(C_5, K_5) = 17$.

*Proof.* Suppose there exists a graph $G$ of order 17 such that $C_5 \not\subset G$ and $\alpha(G) \leq 4$. By Lemma 1 we know that $\delta(G) \geq 4$. Let $u \in V(G)$ be a vertex of degree $\delta(G)$, let $\Gamma$ denote the neighborhood of $u$, and let $W$ denote the set of vertices that remain after $u$ and its neighborhood have been deleted. There are two cases.

*Case (i):* $\delta(G) = 4$. In this case $\langle W \rangle$ is a $C_5$-free graph of order 12 with no four-element independent set. All such graphs are found in [7], and they are listed in the Appendix (§3) of this paper for the reader's convenience. Inspection shows that each one contains a $K_4$ with at least two vertices of degree three. In particular, for each possibility there is a cycle $(w_1, w_2, w_3, w_4, w_1)$ in which $w_1$ and $w_2$ have degree three in $\langle W \rangle$. Since $\delta(G) = 4$, $w_1$ is adjacent to some vertex in $\Gamma$ and so is $w_2$. If $w_1$ and $w_2$ are each adjacent to $v \in \Gamma$ then $(v, w_1, w_4, w_3, w_2, v)$ is a $C_5$ in $G$. If $w_1$ and $w_2$ are adjacent to $v_1$ and $v_2$, respectively, where $v_1 \neq v_2$, then $(u, v_1, w_1, w_2, v_2, u)$ is a $C_5$ in $G$. In either case, we have obtained the desired contradiction.

*Case (ii):* $\delta(G) \geq 5$. In this case $\alpha(G) \geq \Delta(G) \geq 5$ by Lemma 2, a contradiction. □

**Lemma 4.** $r(C_6, K_5) = 21$.

*Proof.* Suppose there exists a graph $G$ of order 21 such that $C_6 \not\subset G$ and $\alpha(G) \leq 4$. Let $V(G) = \{v_1, v_2, \dots, v_{21}\}$. By Lemma 1, $\delta(G) \geq 5$. Also, $r(K_1 + P_4, K_5) = 19$ [5] and $r(C_6, K_4) = 16$, so we may assume that $v_1$ is adjacent to each vertex of the path $(v_2, v_3, v_4, v_5)$, and $I \stackrel{\text{def}}{=} \{v_6, v_7, v_8, v_9\}$ is an independent set. It is easy to check that since $C_6 \not\subset G$, no vertex in $\{v_6, v_7, \dots, v_{21}\}$ is adjacent to two or more vertices of $\{v_2, v_3, v_4, v_5\}$. [If $w$ is adjacent to $v_2$ and $v_3$ then $(w, v_2, v_1, v_5, v_4, v_3, w)$ is a $C_6$ in $G$, if $w$ is adjacent to $v_2$ and $v_4$ then $(w, v_2, v_3, v_1, v_5, v_4, w)$ is a $C_6$ in $G$, and so on.] Since $\alpha(G) \leq 4$ each vertex of $V(G) \setminus I$ is adjacent to at least one vertex of $I$. In view of these two facts, we may assume $\{v_2v_6, v_3v_7, v_4v_8, v_5v_9\} \subset E(G)$. No vertex in $\{v_{10}, \dots, v_{21}\}$ is adjacent to two or more vertices of $I$; otherwise, $G$ contains a $C_6$. Consider $v_6$. Note that $v_1v_6 \notin E(G)$; otherwise $(v_1, v_5, v_4, v_3, v_2, v_6, v_1)$ is a $C_6$ in $G$. Since $\delta(G) \geq 5$ we may assume that $v_6v_j \in E(G)$ for $10 \leq j \leq 13$. Note that $\{v_6, v_{10}, v_{11}, v_{12}, v_{13}\}$ spans a complete subgraph; if $v_iv_j \notin E(G)$ for some $\{i, j\} \subset \{10, 11, 12, 13\}$, then $\{v_7, v_8, v_9, v_i, v_j\}$ is a five-element independent set in $G$. Now the argument can be repeated, except instead of simply containing $K_1 + P_4$, we may assume that the subgraph induced by $\{v_1, v_2, \dots, v_5\}$ is complete. Then either some $i \leq 5$ makes $\{v_i, v_6, v_7, v_8, v_9\}$ a five-element independent set in $G$ or else some $v_j \in I$ is adjacent to two or more vertices of $\{v_1, v_2, \dots, v_5\}$ yielding a $C_6$ in $G$, a contradiction. □

The following lemma provides tools which will be used repeatedly in the remaining proofs. Parts (a) and (b) were used in [1] and parts (c) and (d) appear in [9].

**Lemma 5.** *Suppose $G$ contains the cycle $(u_1, u_2, \ldots, u_{n-1}, u_1)$ of length $n-1$ but no cycle of length $n$. Let $X \subseteq V(G) \setminus \{u_1, u_2, \ldots, u_{n-1}\}$. Then*

*(a) No vertex $x \in X$ is adjacent to two consecutive vertices on the cycle.*

*(b) If $x \in X$ is adjacent to $u_i$ and $u_j$ then $u_{i+1}u_{j+1} \notin E(G)$.*

*(c) If $x \in X$ is adjacent to $u_i$ and $u_j$ then no vertex $x' \in X$ is adjacent to both $u_{i+1}$ and $u_{j+2}$.*

*(d) Suppose $\alpha(G) = m-1$ where $m \leq (n+3)/2$, and $\{x_1, x_2, \ldots, x_{m-1}\} \subset X$ is an $(m-1)$-element independent set. Then no member of this set is adjacent to $m-2$ or more vertices on the cycle.*

*Proof.* (a) Obvious.

(b) If $x \in X$ is adjacent to $u_i$ and $u_j$ where $u_{i+1}u_{j+1} \in E(G)$ then

$$(u_i, x, u_j, u_{j-1}, \ldots, u_{i+1}, u_{j+1}, \ldots, u_{i-1}, u_i)$$

is a $C_n$ in $G$, a contradiction.

(c) If $x$ is adjacent to $u_i$ and $u_j$ and $x'$ is adjacent to $u_{i+1}$ and $u_{j+2}$ then

$$(u_i, x, u_j, u_{j-1}, \ldots, u_{i+1}, x', u_{j+2}, \ldots, u_{i-1}, u_i)$$

is a $C_n$ in $G$, a contradiction.

(d) First notice as did Bondy and Erdős that no $x \in X$ can be adjacent to $m-1$ or more vertices of the cycle. For, if $1 \leq j_1 < j_2 < \cdots < j_{m-1} \leq n-2$ and $x \in X$ satisfies $xu_j \in E(G)$ for all $j \in J = \{j_1, j_2, \ldots, j_{m-1}\}$, then in view of (a) and (b) we see that $\{x\} \cup \{u_{j+1} | \ j \in J\}$ is an $m$-element independent set. Now suppose that $1 \leq k_1 < k_2 < \cdots < k_{m-2} \leq n-3$ and $x \in \{x_1, x_2, \ldots, x_{m-1}\}$ satisfies $xu_k \in E(G)$ for all $k \in K = \{k_1, k_2, \ldots, k_{m-2}\}$. [The condition $n \geq 2m-3$ ensures that there is such an indexing of the vertices on the cycle.] By what was just proved, $x$ is not adjacent to any more vertices on the cycle, in particular $x$ is not adjacent to $v_s$ where $s = k_{m-2}+2$. But $v_s$ is adjacent to some $x' \in \{x_1, x_2, \ldots, x_{m-1}\}$ since otherwise there would be an $m$-element independent set. By (b) we know that $\{u_{k+1} | \ k \in K\}$ is an independent set, and by (c) no member of this set is adjacent to $x'$. It follows that $\{x, x'\} \cup \{u_{k+1} | \ k \in K\}$ is an $m$-element independent set, a contradiction. □

**The Standard Configuration.** To prove that $r(C_n, K_5) = 4(n-1)+1$ for $n \geq 7$, we shall in each case assume to the contrary that there exists a graph $G$ of order $4(n-1)+1$ such that $C_n \not\subset G$ and $\alpha(G) \leq 4$. By Lemma 1, $\delta(G) \geq n-1$. By induction, $r(C_{n-1}, K_5) = 4(n-2)+1$. Hence we may assume that $(u_1, u_2, \ldots, u_{n-1}, u_1)$ is a cycle of length $n-1$ in $G$ and, disjoint from this cycle, there is a four-element independent set $I = \{v_1, v_2, v_3, v_4\}$. Let $C = V(C_{n-1}) = \{u_1, u_2, \ldots, u_{n-1}\}$ denote the set of vertices on the cycle, and let $W = V(G) \setminus (C \cup I) = \{w_1, w_2, \ldots, w_{3n-6}\}$ denote the set of vertices disjoint from $C \cup I$. Since $\alpha(G) \leq 4$ each vertex in $C$ is adjacent to at least one vertex in $I$. In view of part (d) of Lemma 5 (with $m = 5$), no member of $I$ is adjacent to 3 or more vertices on the cycle. Thus the set of edges $E(C, I) = \{uv|\ u \in C,\ v \in I\}$ satisfies $|C| \leq |E(C, I)| \leq 8$. If $v \in I$ is adjacent to $u_i$ and $u_j$ and these two vertices have no other neighbors in $I$ then $u_i u_j \in E(G)$; otherwise, $u_i, u_j$ and the three members of $I \setminus \{v\}$ yield a five-element independent set. Note that each vertex in $I$ is adjacent to at least $n - 3$ vertices in $W$. Since $4(n-3) > 3n - 6$, we may assume (if needed) that there are two vertices in $I$ that are commonly adjacent to some vertex $w \in W$. The structure just described will be called the *standard configuration.*

**Lemma 6.** $r(C_7, K_5) = 25$.

*Proof.* Assume the standard configuration. Then $6 \leq |E(C, I)| \leq 8$. The proof is divided into two parts. The first part deals with the possibility $7 \leq |E(C, I)| \leq 8$ and the second part with $|E(C, I)| = 6$.

*Part I:* $7 \leq |E(C, I)| \leq 8$. Note that each vertex in $I$ is adjacent to at least one vertex in $C$. If not, then some other vertex in $I$ is adjacent to at least $\lceil 7/3 \rceil = 3$ vertices in $C$, contradicting part (d) of Lemma 5 (with $m = 5$). In case (i) below, we use the prerogative of assuming that $v_1$ and $v_2$ are commonly adjacent to some $w \in W$. We may assume that $v_1$ is adjacent to two vertices in $C$. There are two cases.

*Case (i):* $v_1$ is adjacent to $u_1$ and $u_3$. Note that $u_2 u_4 \notin E(G)$ and $u_2 u_6 \notin E(G)$, both by part (b) of Lemma 5. Also $u_4 v_2 \notin E(G)$; otherwise $(w, v_1, u_1, u_2, u_3, u_4, v_2, w)$ is a $C_7$ in $G$. In the same way, $u_6 v_2 \notin E(G)$. We now make two claims.

*Claim 1:* $u_5 v_2 \notin E(G)$. Suppose $u_5 v_2 \in E(G)$. Then $u_2 v_2 \notin E(G)$ by part (c) of Lemma 5 and $u_4 u_6 \notin E(G)$ as well; otherwise $(w, v_1, u_1, u_6, u_4, u_5, v_2, w)$ is a $C_7$ in $G$. In this case, $\{u_2, u_4, u_6, v_1, v_2\}$ is a five-element independent set in $G$, a contradiction.

*Claim 2:* $u_2 v_2 \in E(G)$. Suppose $u_2 v_2 \notin E(G)$. Then $u_4 u_6 \in E(G)$ since otherwise $\{u_2, u_4, u_6, v_1, v_2\}$ is a five-element independent set in $G$. Then $u_1 v_2 \notin E(G)$; otherwise $(w, v_1, u_3, u_4, u_6, u_1, v_2, w)$ is a $C_7$ in $G$. In the same way $u_3 v_2 \notin E(G)$. Then $u v_2 \notin E(G)$ for all $u \in C$, a contradiction.

In view of part (a) of Lemma 5 and previously established facts, this means that $v_2$ is adjacent to precisely one vertex in $C$. Hence if $|E(C, I)| = 8$, we have already reached a contradiction. Now we may assume that $u_4$ and $u_6$ are both adjacent to $v_3$ and $u_5$ is adjacent to $v_4$. Then $u_4 u_6 \in E(G)$; otherwise $\{u_2, u_4, u_6, v_1, v_4\}$ is a five-element independent set in $G$. Note that $u_2 v_4 \notin E(G)$ by part (c) of Lemma 5. Also, $u_1 v_4 \notin E(G)$ and $u_3 v_4 \notin E(G)$, both by part (b) of Lemma 5. It follows that $v_4$ is adjacent to precisely one vertex on the cycle, so $|E(C, I)| = 2 + 1 + 2 + 1 = 6$, a contradiction. This completes the proof in case (i).

*Case (ii):* $v_1$ is adjacent to $u_1$ and $u_4$. In this case, we do not make use of the assumption that $v_1$ and $v_2$ are commonly adjacent to $w \in E(G)$. This means that the three vertices $v_2, v_3, v_4$ are on an equal footing. A simple argument, sketched below, shows that a second vertex, which we may take to be $v_2$, is adjacent to $u_2$ and $u_5$ or to $u_3$ and $u_6$. [If we deny this conclusion and use part (c) of Lemma 5, we find that if $v \in \{v_2, v_3, v_4\}$ is adjacent to two vertices in $C$, then one of those vertices must be $u_1$ or $u_4$. For each such $v$ there is an extra edge in $E(C, I)$ over the six that are required by the fact that each vertex in $C$ is adjacent to at least one vertex in $I$. Suppose there are $k$ such vertices. By the observation just made, $|E(C, I)| \geq 6 + k$. On the other hand, the appropriate degree sum for vertices in $I$ yields $|E(C, I)| = 2(k+1) + (3-k) = 5 + k$.] Hence there are two subcases.

*Subcase (a): $v_2$ is adjacent to $u_2$* and $u_5$. Then $u_3u_6 \notin E(G)$ by part (b) of Lemma 5. For $v \in \{v_3, v_4\}$, either $u_3v \in E(G)$ or $u_6v \in E(G)$; otherwise $\{u_3, u_6, v_1, v_2, v\}$ is a five-element independent set in $G$. If $u_3v \in E(G)$ then $u_1v \notin E(G)$ and $u_4v \notin E(G)$, by parts (c) and (a), respectively, of Lemma 5. If $u_6v \in E(G)$ then $u_1v \notin E(G)$ and $u_4v \notin E(G)$ by parts (a) and (c), respectively, of Lemma 5. In view of this, $\{u_1, u_4, v_2, v_3, v_4\}$ is a five-element independent set in $G$, a contradiction.

*Subcase (b): $v_2$ is adjacent to $u_3$* and $u_6$. The proof is similar to that of subcase (a). First $u_2u_5 \notin E(G)$ by part (b) of Lemma 5. Then for $v \in \{v_3, v_4\}$ either $u_2v \in E(G)$ or $u_5v \in E(G)$. Finally, for $u_1v \notin E(G)$ and $u_4v \notin E(G)$ for $v \in \{v_3, v_4\}$, so $\{u_1, u_4, v_1, v_3, v_4\}$ is a five-element independent set in $G$, a contradiction. This completes the proof in Part I.

*Part II:* $|E(C, I)| = 6$. In this part, each vertex in $C$ is adjacent to precisely one vertex in $I$, so if $v \in I$ is adjacent to $u_i$ and $u_j$ then $u_iu_j \in E(G)$. Do not assume that $v_1$ and $v_2$ are both adjacent to $w \in W$, only that some pair $v_i, v_j \in I$ have this property. Without loss of generality, $v_1$ is adjacent to two vertices in $C$. There are two cases.

*Case (i): $v_1$ is adjacent to $u_1$ and $u_4$.* Then we may assume that $u_2$ is adjacent to $v_2$. In view of parts (a), (b), and (c) of Lemma 5 and the fact that each vertex in $C$ is adjacent to precisely one vertex in $I$, it is clear that $u_iv_2 \notin E(G)$ for $i \neq 2$. In the same way, we may assume that $u_3$ is adjacent to $v_3$ and then find that $u_iv_3 \notin E$ for $i \neq 3$. Then we may assume that $u_5$ is adjacent to $v_4$. Finally, however, $v_6$ cannot be adjacent to any vertex in $I$, a contradiction.

*Case (ii): $v_1$ is adjacent to $u_1$ and $u_3$.* Then $u_1u_3 \in E(G)$. We may assume that $u_2$ is adjacent to $v_2$. As before, we then find that $u_iv_2 \notin E(G)$ for $i \neq 2$. Then, in the only acceptable configuration, $u_4$ and $u_6$ are both adjacent to $v_3$, $u_4u_6 \in E(G)$, $u_5v_4 \in E(G)$ and $u_iv_4 \notin E(G)$ for $i \neq 5$. Now we use the fact that there are two vertices $v_i, v_j \in I$ that are both adjacent to $w \in W$. If $v_1$ and $v_3$ are both adjacent to $w$ then $(w, v_1, u_1, u_2, u_3, u_4, v_3, w)$ is a $C_7$ in $G$. If $v_1$ and $v_4$ are both adjacent to $w$ then $(w, v_1, u_1, u_3, u_4, u_5, v_4, w)$ is a $C_7$ in $G$. If $v_2$ and $v_4$ are adjacent to $w$ then $(w, v_2, u_2, u_3, u_4, u_5, v_4, w)$ is a $C_7$ in $G$. Hence, by symmetry, we may assume that $v_1$ and $v_2$ are both adjacent to $w \in W$. Let $Z = \{u_1, \ldots, u_6, v_1, \ldots, v_4, w\}$ and $Z' = V(G) \setminus Z$.

As one may readily verify, for each vertex $z \in Z \setminus \{v_1, v_2, w\}$ there is a path of length six from $w$ to $z$. Also for each $z \in Z \setminus \{u_4, u_6, v_4\}$ there is a path of length six from $u_5$ to $z$. Since $C_7 \not\subset G$, the degrees of $u_5, v_2, v_3, w$ in $\langle Z \rangle_G$ are $3, 2, 2, 2$, respectively. Since $\delta(G) \geq 6$ there are at least $3 + 4 + 4 + 4 = 15$ edges joining $S \stackrel{\text{def}}{=} \{u_5, v_2, v_3, w\}$ and $Z'$. Since $|Z'| = 14$, there must be two vertices in $S$ that are adjacent to the same $w' \in Z'$. Finally, the following path system shows that any two vertices in $S$ are joined by a path of length five in $\langle Z \rangle_G$:

$$\begin{array}{ll} (u_5, u_4, u_3, u_1, u_2, v_2), & (u_5, u_4, u_3, u_1, u_6, v_3), \\ (u_5, u_4, u_3, u_2, v_2, w), & (v_2, u_2, u_1, u_3, u_4, v_3), \\ (v_2, u_2, u_1, u_3, v_1, w), & (v_3, u_4, u_3, u_2, v_2, w). \end{array}$$

Since there are two vertices in $S$ that are both adjacent to $w' \in Z'$, this gives a $C_7$ in $G$, a contradiction. □

**Lemma 7.** $r(C_8, K_5) = 29$.

*Proof.* Assume the standard configuration. The edge count $7 = |C| \leq |E(C, I)| \leq 8$ gives two cases for consideration.

*Case (i):* $|E(C, I)| = 7$. In this case, each vertex in $C$ is adjacent to exactly one vertex in $I$, one (exceptional) vertex in $I$ is adjacent to only one vertex in $C$, and the other three are each adjacent to two vertices on the cycle. We may assume that $v_1$ is not the exceptional vertex. Let $N$ denote the neighbors of $v_1$ in $C$. By symmetry, there are two subcases.

*Subcase (a):* $N = \{u_1, u_3\}$. Then $u_1u_3 \in E(G)$, and we may assume that $u_2$ is adjacent to $v_2$. It is easily checked that there is a path of order eight joining $v_2$ and $u_i$ for $i = 4, 5, 6, 7$. Since there would be a $C_8$ otherwise, we may assume that $u_iv_2 \notin E(G)$ for $i = 1, 3, 4, 5, 6, 7$, so $v_2$ must be the exceptional vertex. Then we may assume that $v_3$ is adjacent to $u_4$ and $u_6$, and that $v_4$ is adjacent to $u_5$ and $u_7$, so $u_5u_7 \in E(G)$. But this violates part (b) of Lemma 5.

*Subcase (b):* $N = \{u_1, u_4\}$. Then $u_1u_4 \in E(G)$, and we may assume that that $u_2$ is adjacent to $v_2$ and $u_3$ is adjacent to $v_3$. Note that there is a path of order eight joining $v_i$ and $u_j$ for $i = 2, 3$ and $j = 5, 6, 7$. But $v_2$ and $v_3$ are not both exceptional, so we have a contradiction.

*Case (ii):* $|E(C, I)| = 8$. In this case, one (exceptional) vertex in $C$ is adjacent to two vertices in $I$, and each vertex in $I$ is adjacent to two vertices in $C$. As noted earlier, we may assume that there is a vertex $w \in W$ that is adjacent to both $v_1$ and $v_2$. Again let $N$ denote the neighbors of $v_1$ in $C$.

*Subcase (a):* $N = \{u_1, u_3\}$. Note that there is a path of order eight joining $v_2$ and $u_i$ for $i = 4, 5, 6, 7$, so $v_2$ cannot be adjacent to $u_4, u_5, u_6$ or $u_7$. Also $v_2$ cannot be adjacent to $u_1$ and $u_2$ or to $u_2$ and $u_3$ by part (a) of Lemma 5. Finally, $v_2$ cannot be adjacent to both $u_1$ and $u_3$ since there is only one exceptional vertex in $C$. Hence there do not exist two vertices on the cycle that can serve as neighbors of $v_2$, a contradiction.

*Subcase (b):* $N = \{u_1, u_4\}$. Note that there is a path of order eight joining $v_2$ and $u_i$ for $i = 1, 4, 5, 7$. Hence we may assume that $v_2$ is adjacent to $u_2$ and $u_6$. However, this violates part (c) of Lemma 5.

Since a contradiction arises in each subcase, the lemma is proved. □

**Lemma 8.** $r(C_9, K_5) = 33$.

*Proof.* Assume the standard configuration. The edge count $8 = |C| \leq |E(C, I)| \leq 8$ requires each vertex in $C$ to be adjacent to exactly one vertex of $I$ and each vertex in

$I$ to be adjacent to exactly two vertices in $C$. We may assume that there is a vertex $w \in W$ that is adjacent to both $v_1$ and $v_2$. Let $N = \{u_i, u_j\}$ denote the neighbors of $v_1$ on the cycle. Since there is no five-element independent set, $u_i u_j \in E(G)$. By symmetry, there are three cases.

*Case (i):* $N = \{u_1, u_3\}$. It is easily checked that for $4 \leq i \leq 8$ there is a path of order seven joining $v_1$ and $u_i$. The paths $(v_1, u_1, u_8, u_7, u_6, u_5, u_4)$ and $(v_1, u_3, u_1, u_8, u_7, u_6, u_5)$ serve for $i = 4$ and $i = 5$, respectively, and their counterparts by symmetry take care of $i = 8$ and $i = 7$. The required path for $i = 6$ may be taken to be $(v_1, u_1, u_2, u_3, u_4, u_5, u_6)$. Hence there do not exist two vertices on the cycle that can serve as neighbors of $v_2$.

*Case (ii):* $N = \{u_1, u_4\}$. In this case for $5 \leq i \leq 8$ there is a path of order seven joining $v_1$ and $u_i$. The paths $(v_1, u_4, u_1, u_8, u_7, u_6, u_5)$ and $(v_1, u_1, u_2, u_3, u_4, u_5, u_6)$ serve for $i = 5$ and $i = 6$, respectively, and symmetric counterparts take care of $i = 8$ and $i = 7$. Therefore $v_2$ cannot be adjacent to any of the vertices $u_5, u_6, u_7, u_8$. By part (a) of Lemma 5, $v_2$ cannot be adjacent to $u_2$ and $u_3$. Hence there do not exist two vertices on the cycle that can serve as the neighbors of $v_2$.

*Case (iii):* $N = \{u_1, u_5\}$. In this case, there is a path of order seven joining $v_1$ to $u_i$ for $i = 2, 4, 6, 8$, so the neighbors of $v_2$ on the cycle must be $u_3$ and $u_7$. Without loss of generality, $u_2$ is adjacent to $v_3$, and by symmetry the neighbors of $v_3$ on the cycle are either $u_2$ and $u_4$ or $u_2$ and $u_6$. In the first instance, $u_2 u_4 \in E(G)$ and $(v_1, u_1, u_8, u_7, v_2, u_3, u_2, u_4, u_5, v_1)$ is a $C_9$ in $G$. In the second, $u_2 u_6 \in E(G)$ and $(v_1, u_1, u_8, u_7, u_6, u_2, u_3, u_4, u_5, v_1)$ is a $C_9$ in G.

Since a contradiction arises in each case, the proof is complete. □

*Completion of the proof of Theorem 1.* For $n \geq 10$, the edge count $n - 1 = |C| \leq |E(C, I)| \leq 8$ gives an immediate contradiction. □

# 3 Appendix - Possible Induced Subgraphs $\langle W \rangle$ for Case (i) in Lemma 3

Here we give the promised collection of graphs of order 12 that contain no $C_5$ and have independence number 3.

**Proposition.** *If $G$ is a graph of order twelve such that $C_5 \not\subseteq G$ and $\alpha(G) = 3$ then $G$ is isomorphic to $3K_4$ or to one of the five graphs shown below, obtained by adding appropriate edges to $3K_4$.*

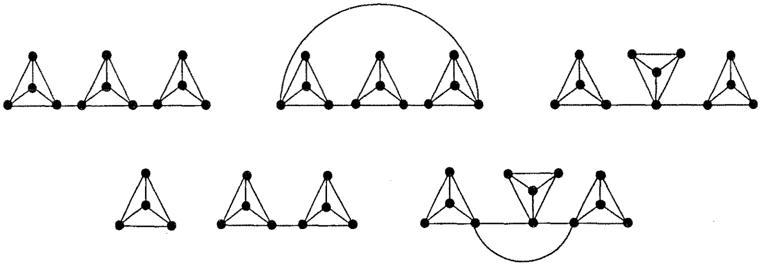

FIGURE 1. Graphs of order twelve with $C_5 \not\subseteq G$ and $\alpha(G) = 3$.

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