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Diagonal Ramsey numbers in multipartite graphs related to stars

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Abstract

Abstract: Let the star in n vertices, namely $K_{1,n-1}$ be denoted by S_n. If every two coloring of the edges of a complete balanced multipartite graph $K_{j\times s}$ there is a copy of S_n in the first color or a copy of S_m in the second color, then we will say $K_{j\times s} \to (S_n, S_m)$. The size Ramsey multipartite number $m_j(S_n, S_m)$ is the smallest natural number s such that $K_{j\times s} \to (S_n, S_m)$. In this paper, we obtain the **exact** values of the size Ramsey numbers $m_j(S_n, S_m)$ for $n, m \geq 3$ and $j \geq 3$.

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1. Introduction

In this paper we concentrate on simple graphs without loops and multiple edges. Let the complete multipartite graph having j uniform sets of size s be denoted by $K_{j\times s}$ and the complete bipartite graph on $n+m$ vertices be denoted by $K_{n,m}$. Given, three graphs K_N, G and H, we say that $K_N \to (G, H)$ if K_N is colored by two colors red and blue and it contains a copy of G(in the first color) or a copy of H(in the second color). Using this notation we define the classical Ramsey number $r(n, m)$ as the smallest integer N such that $K_N \to (K_n, K_m)$. However, even in the case of diagonal classical Ramsey numbers $r(n, n)$ almost nothing significant is known beyond the case $n = 5$ (see Radziszowski et al 2014 for a survey).

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In the decades that followed there are several interesting variations that have originated from these classical Ramsey number. One obvious variation is the case of size Ramsey numbers build up mainly by Erdös, Faudree, Rousseau and Shelph (see Erdös et al 1978 and Faudree et al 1975). Another variation introduced recently by Buger et al 2004 and Syafrizal et al 2005, is the concept of balanced multipartite Ramsey numbers. This concept is based on exploring the two colorings of multipartite graphs $K_{j\times s}$ instead of the complete graph. Formally define size Ramsey multipartite number $m_j(G, H)$ as the smallest natural number s such that $K_{j\times s} \to (G, H)$. However, currently there are very few known multipartite Ramsey numbers between pairs of graphs and pairs of classes of graphs other than the ones introduced initially by Syafrizal et al 2005 and 2009, Lusiani et al 2015 and Jayawardene et al 2016 and 2017.

Notation

Given a graph $G = (V, E)$ with the *order* of the graph is denoted by $|V(G)|$ and the *size* of the graph is denoted by $|E(G)|$. For a vertex v of a graph G, the *neighborhood* of v is denoted by $N(v)$ and is defined as the set of vertices adjacent to v. Further the cardinality of this set, denoted $d(v)$, is defined as the *degree* of v. We say that a graph G is a k-regular graph if $d(v) = k$ for all $v \in V(G)$. Given a red-blue coloring of $K_{j\times s} = H_R \oplus H_B$. The red degree and blue degree of any vertex v belonging to $V(K_{j\times s}) = \{v_{k,i} | \ 0 \le i \le s-1, 0 \le k \le j-1\}$ denoted by $d_R(v)$ and $d_B(v)$ respectively, are defined as the degree of vertex v in H_R and H_B respectively.

Given $w \ge 2$, $0 \le i \le w-1$ and $0 < c \le w-1$, define $\sigma_{c,w}(i) = \{a_1\} \cup \{a_2\}$, $\sigma^+_{c,w}(i) = \{a_1\}$, $\sigma^-_{c,w}(i) = \{a_2\}$ and $B_{0,w}(i) = \phi$ and if $k > 0$, $B_{k,w}(i) = \cup_{c=1}^{k} \sigma_{c,w}(i)$ where $a_1 = (i+c) \mod k$ and $a_2 = (i-c) \mod k$.

2. Some Lemmas

In all the following lemmas assume $d > 0$ as the results are trivially true when $d = 0$.

Lemma 2.1. *There exists a regular induced subgraph of degree d of $K_{j\times s}$ on the vertex set $V(K_{j\times s})$ provided that d is even, j is odd and s is odd.*

Proof. Let $d = 2k_1(j-1)+2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \le s-1$ and $0 < 2k_2 \le j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \ne i$. **b)** If $r = l$ and $p \in B_{k_2,j}(i)$.

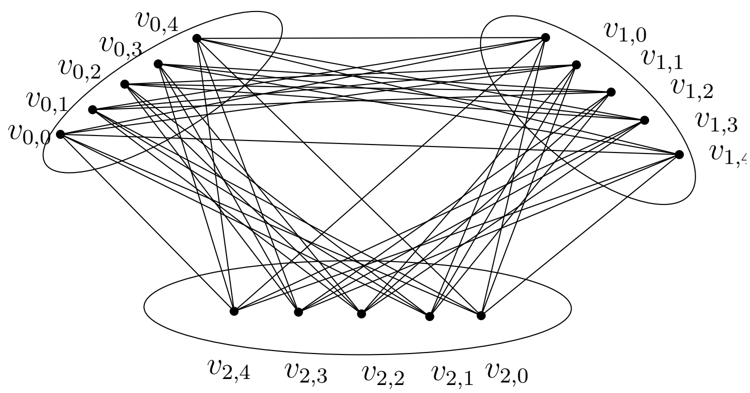

Figure 1: In the case when $d = 6 = 2k_1(j-1) + 2k_2 = (2 \times 1)(3-1) + 2 \times 1$

We know that $K_{j \times s}$ consists of j partite sets of size s. Given $v_{i,l}$ $0 \leq i \leq j-1$, $0 \leq l \leq s-1$, the set $\{v_{p,r} |\ p \neq i$ and $r \in B_{k_1,s}(l)\}$ will represent the vertices not belonging to the i^{th} partite set (denoted by V_i) that are at most $2k_1$ distance apart inside a partite set (with respect to the second coordinate), as illustrated in the following figure.

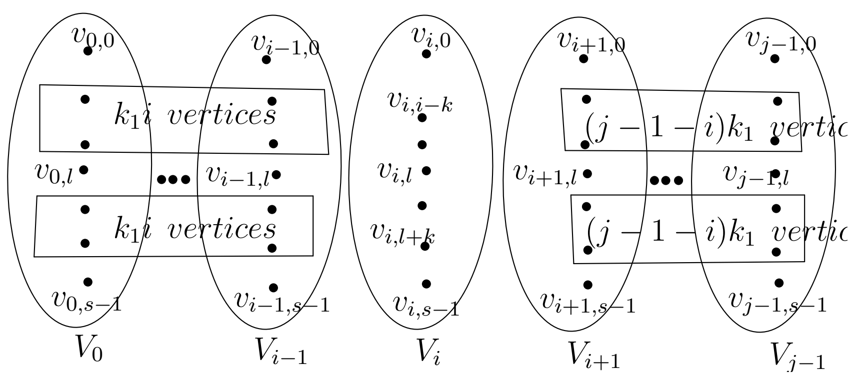

Figure 2: The set consisting of $2k_1(j-1)$ vertices corresponding to part(a), namely $\{v_{p,r} |\ p \neq i$ and $r \in B_{k_1,s}(l)\}$

More precisely, it will consist of the vertices

$v_{0,l-k_1}, \ldots, v_{i-1,l-k_1}, v_{i+1,l-k_1}, \ldots, v_{j-1,l-k_1}$
$v_{0,l-k_1+1}, \ldots, v_{i-1,l-k_1+1}, v_{i+1,l-k_1+1}, \ldots, v_{j-1,l-k_1+1}$
...
$v_{0,l-1}, \ldots, v_{i-1,l-1}, v_{i+1,l-1}, \ldots, v_{j-1,l-1}$
$v_{0,l+1}, \ldots, v_{i-1,l+1}, v_{i+1,l+1}, \ldots, v_{j-1,l+1}$
...
$v_{0,l+k_1-1}, \ldots, v_{i-1,l+k_1-1}, v_{i+1,l+k_1-1}, \ldots, v_{j-1,l+k_1-1}$
$v_{0,l+k_1}, \ldots, v_{i-1,l+k_1}, v_{i+1,l+k_1}, \ldots, v_{j-1,l+k_1}$

That is such a set consists of $2k_1(j-1)$ vertices.

Similarly, given $v_{i,l}$ where $0 \leq i \leq j-1$ and $0 \leq l \leq s-1$ the set $\{v_{p,r} | \, r \neq l$ and $p \in B_{k_2,j}(i)\}$ will represent the vertices not belonging to the i^{th} partite set(denoted by V_i) that are at most $2k_2$ distance apart between partite sets (with respect to the first coordinate), as illustrated in the following figure. More precisely, it will consist of the vertices

$$v_{i-k_2,l}, \ldots, v_{i-1,l}, v_{i+1,l}, \ldots, v_{i+k_2,l}$$

That is such a set consists of $2k_2$ vertices.

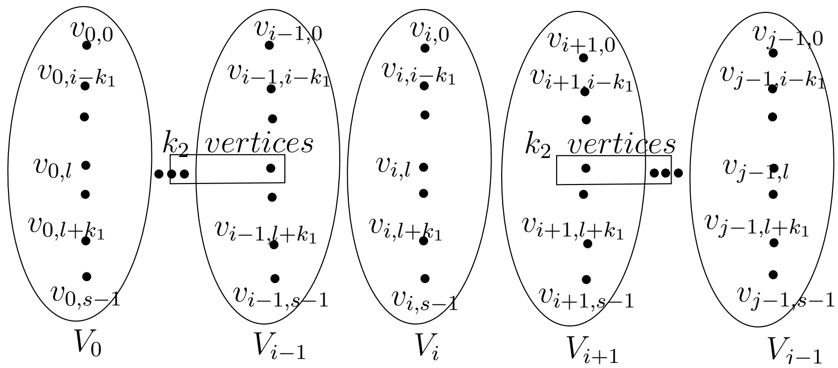

Figure 3: The set consisting of $2k_2$ vertices corresponding to part(b), namely $\{v_{p,r} | \, r = l$ and $p \in B_{k_2,j}(i)\}$

Thus, by the above definition, part (a) will represent $2k_1(j-1)$ vertices adjacent to $v_{i,l}$ belonging to $V_0, V_1, V_2, \ldots, V_{i-1}, V_{i+1}, \ldots V_{j-1}$ and part (b) will represent another $2k_2$ vertices adjacent to $v_{i,l}$ belonging to $V_{i-k_2}, V_{i-k_2+1}, \ldots, V_{i-1}, V_{i+1}, \ldots, V_{i+k_2-1}, V_{i+k_2}$.

Therefore, the degree of $v_{i,l}$ will be equal to $2k_1(j-1) + 2k_2$. Moreover, we get that if any vertex of V_j is adjacent to a vertex of V_k by the rule (a) (or rule(b)) then the exact same rule will dictate that particular vertex of V_k also to be adjacent to the exact same vertex of V_j. Thus, the generated graph is well defined.

Next, let $d = (2k_1+1)(j-1) + 2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-3$ and $0 < 2k_2 \leq j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If there exists w such that $4w = (j-1+2k_2)$ and $r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)$.

c) If there exists w such that $(j-1+2k_2) - 4w = 2$ and $r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)$ or else $r = l$ and $p \in B_{1,j}(i)$.

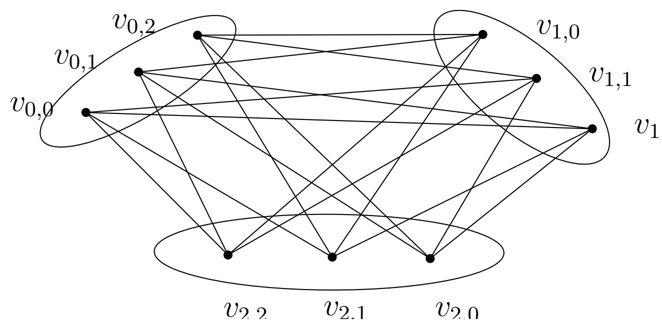

Figure 4: In the case when $d = 4 = (2k_1 + 1)(j - 1) + 2k_2 = (2 \times 0 + 1) \times (3 - 1) + 2 \times 1$

It should be noted that the vertex sets of part (b) and part (c) are disjoint and that $j - 1 + 2k_2$ is even as j is odd. Therefore, given $v_{i,l}$, it will be either adjacent the vertices corresponding to part (a) and part (b) or else adjacent the vertices corresponding to part (a) and part (c) according to whether $4w = (j - 1 + 2k_2)$ or else $(j - 1 + 2k_2) - 4w = 2$, respectively.

As illustrated in figure 2, the set generated by part (a) namely, $\{v_{p,r} | \ p \neq i$ and $r \in B_{k_1,s}(l)\}$ will consist of $2k_1(j - 1)$ vertices.

Similarly, given $v_{i,l}$ the set generated by part (b), will represent the vertices belonging to $V_{i-w}, ...,$ $V_{i-1}, V_{i+1}, ...V_{i+w}$ sets, that are at most $2(k_1 + 1)$ distance apart inside a given partite set (i.e., with respect to the second coordinate) since $(l + (k_1 + 1)) - (l - (k_1 + 1)) = 2(k_1 + 1)$. Moreover, it will consist of the vertices

$$v_{i-w,l-(k_1+1)}, \ldots, v_{i-1,l-(k_1+1)}, v_{i+1,l-(k_1+1)}, \ldots v_{i+w,l-(k_1+1)}$$
$$v_{i-w,l+(k_1+1)}, \ldots, v_{i-1,l+(k_1+1)}, v_{i+1,l+(k_1+1)}, \ldots v_{i+w,l+(k_1+1)}$$

Such a set consists of $4w$ vertices. That is, the set consists of $(j - 1) + 2k_2$ vertices.

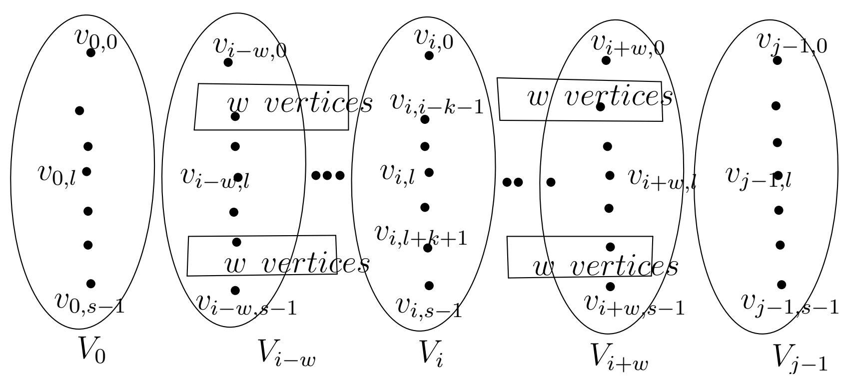

Figure 5: The set consisting of $4w$ vertices corresponding to part (b), namely $\{v_{p,r} | \ r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)\}$

Similarly, given $v_{i,l}$ the set generated by the later part (c), will represent the two vertices belonging to V_{i-1}, V_{i+1} sets, namely $v_{i-1,l-1}$, $v_{i+1,l+1}$. More precisely the set generated by part (c), will consist of the vertices

$v_{i-w,l-(k_1-1)}, \dots, v_{i-1,l-(k_1-1)}, v_{i+1,l-(k_1-1)}, \dots v_{i+w,l-(k_1-1)}$
$v_{i-1,l-1}, v_{i+1,l+1}$
$v_{i-w,l-(k_1+1)}, \dots, v_{i-1,l-(k_1+1)}, v_{i+1,l-(k_1+1)}, \dots v_{i+w,l-(k_1+1)}$

Such a set consists of $4w+2$ vertices. That is, the set consists of $(j-1)+2k_2$ vertices.

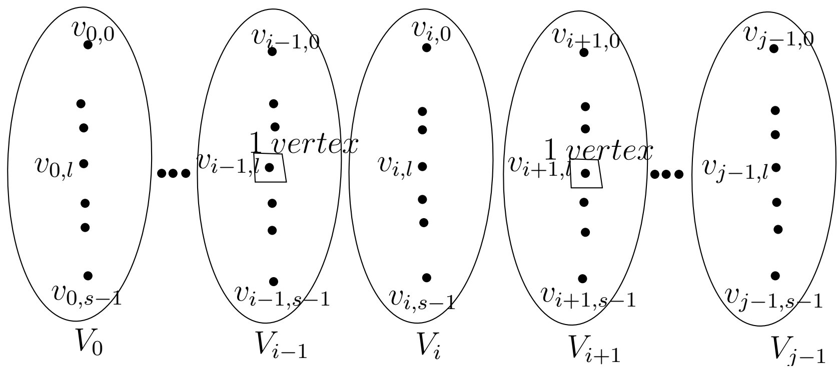

Figure 6: The set consisting of 2 vertices corresponding to later section of part (c), namely $\{v_{p,r}|$ $p \in B_{w,j}(i)$ or else $r = l$ and $p \in B_{1,j}(i)\}$

Therefore, the degree of $v_{i,l}$ will be equal to $2k_1(j-1)+2k_2$ when part (a)+(b) situation arises or when part (a)+(c) situation arises. □

Lemma 2.2. *There exist regular graphs of degree d on $V(K_{j\times s})$ if j is even or s is even.*

Proof. We approach this problem by considering the following three cases.

Case 1) If j is even and s is odd.
If $d = 2k_1(j-1)+2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-1$ and $0 < 2k_2 \leq j-2$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$. **b)** If $r = l$ and $p \in B_{k_2,j}(i)$.

The vertex $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) or part (b) and they are respectively equal to $2k_1(j-1)$ and $2k_2$. Therefore, we get that the degree of $v_{i,l}$ is equal to $2k_1(j-1)+2k_2$ as required.

Let $d = 2k_1(j-1)+2k_2+1$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-1$ and $2k_2 \leq j-2$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$. **b)** If $r = l$ and $p \in B_{k_2,j}(i)$.

c) If $r = l$ and $p \in \sigma_{\frac{j}{2},j}(i)$.

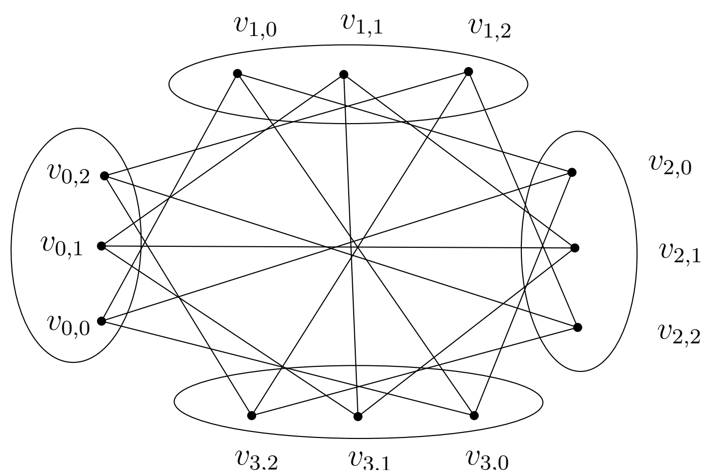

Figure 7: In the case(1) when $d = 3 = 2k_1(j-1) + 2k_2 + 1 = 2 \times 0 \times (4-1) + 2 \times 1 + 1$

The vertex $v_{i,l}$ will be either adjacent the vertices corresponding to part (a), part (b) or part (c) and they are respectively equal to $2k_1(j-1),\ 2k_2$ and one. Therefore, we get that the degree of $v_{i,l}$ is equal to $2k_1(j-1) + 2k_2 + 1$ as required.

Let $d = (2k_1+1)(j-1) + m$ for some non negative integers k_1, k_2 and m such that $2k_1 \leq s-3$ and $0 < m \leq j-1$ where $m = 2k_2$ or $m = 2k_2+1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold
a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.
b) If there exists w such that $w = (j-1+m)$ div 4, $r \in \sigma_{k_1+1,s}(l)$ and $p \in B_{w,j}(i)$.
c) If there exists w such that $(j-1+m) - 4w = 1$, $r = l$ and $p \in \sigma_{\frac{j}{2},j}(i)$.
d) If there exists w such that $(j-1+m) - 4w = 2$, $r = l$ and $p \in \sigma_{1,j}(i)$.
e) If there exists w such that $(j-1+m) - 4w = 3$, $r = l$, $p \in \sigma_{1,j}(i)$ and $p \in \sigma_{\frac{j}{2},j}(i)$ (as j is even).

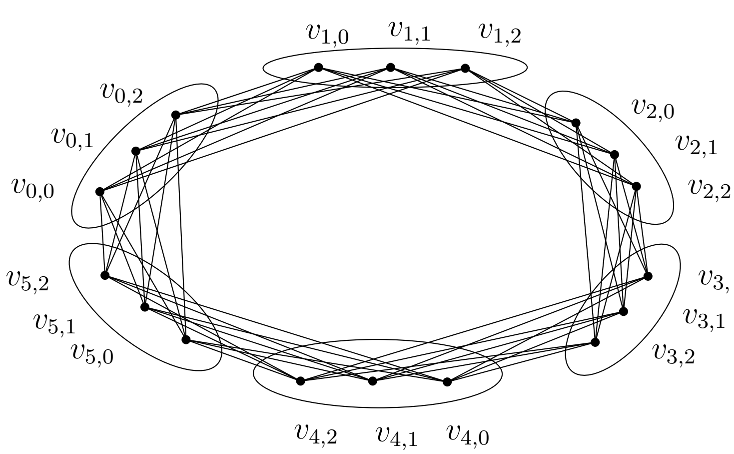

Figure 8: In the case(1) when $d = 6 = (2k_1+1)(j-1) + 2k_2 + 1 = (2 \times 0 + 1)(6-1) + 2 \times 0 + 1$

It should be noted that the vertex sets of part (b), (c), (d) and part (e) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) or (a) and part (d) or else part (a) and part (e) according to whether $4w = j-1+2k_2$, $(j-1+2k_2)-4w = 1$, $(j-1+2k_2)-4w = 2$ or $(j-1+2k_2)-4w = 3$ respectively. In all these scenarios we get $d = (2k_1+1)(j-1)+m$ as required.

Case 2) If j is even and s is even.

Let $d = 2k_1(j-1)+2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-2$ and $0 < k_2 \leq j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 < \frac{j}{2}$, $r = l$ and $p \in B_{k_2,j}(i)$.

c) If $\frac{j}{2} \leq k_2 < j-1$ and $((r = l$ and $p \in B_{\frac{j-2}{2},j}(i))$ or $(r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-2)}{2},j}(i)))$.

d) If $k_2 = j-1$ and $((r = l$ and $p \neq i)$ or $(r \in \sigma_{\frac{s}{2},s}(l)$ and $p \neq i))$.

It should be noted that the vertex sets of part (b), (c) and part (d) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) or else part (a) and part (d) according to whether $k_2 < \frac{j}{2}$, $\frac{j}{2} \leq k_2 < j-1$ or $k_2 = j-1$ respectively. In all these scenarios we get $d = 2k_1(j-1)+2k_2$ as required.

Let $d = 2k_1(j-1)+2k_2+1$ for some non negative integers k_1 and k_2 where such that $2k_1 \leq s-2$ and $k_2 < j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 \leq \frac{j-2}{2}$, $r = l$ and $(p \in B_{k_2,j}(i)$ or $p \in \sigma_{\frac{j}{2},j}(i))$.

c) If $k_2 \geq \frac{j}{2}$ and $((r = l$ and $p \neq i)$ or $(r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-2)}{2},j}(i)))$.

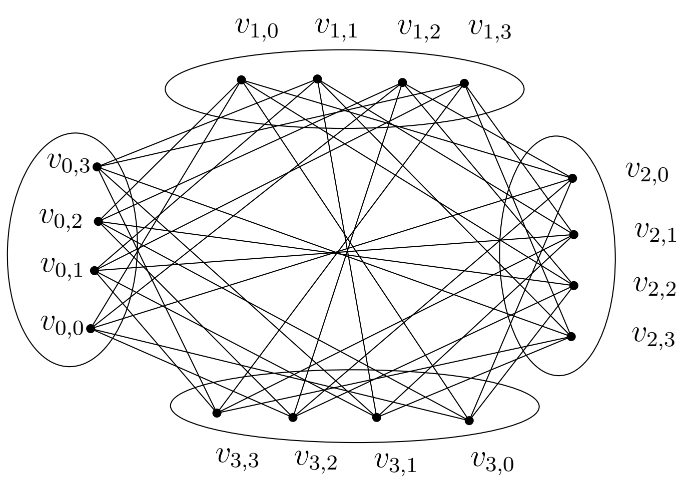

Figure 9: In the case(2) when $d = 5 = 2k_1(j-1)+2k_2+1 = 2 \times 0 \times (4-1)+2 \times 2+1$

It should be noted that the vertex sets of part (b) and part (c) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) according to whether $k_2 \leq \frac{j-2}{2}$ or $k_2 \geq \frac{j}{2}$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2 + 1$ as required.

Case 3) If j is odd and s is even.

Let $d = 2k_1(j-1) + 2k_2$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-2$ and $0 < k_2 \leq j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 < \frac{j}{2}$, $r = l$ and $p \in B_{k_2,j}(i)$

c) If $k_2 \geq \frac{j}{2}$ and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-1)}{2},j}(i)$)).

It should be noted that the vertex sets of part (b) and part (c) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) according to whether $k_2 < \frac{j}{2}$ or $k_2 \geq \frac{j}{2}$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2$ as required.

Case 4) If j is odd and s is even.

Let $d = 2k_1(j-1) + 2k_2 + 1$ for some non negative integers k_1 and k_2 such that $2k_1 \leq s-2$ and $0 \leq k_2 < j-1$. Construct a graph by connecting the vertices $v_{i,l}$ and $v_{p,r}$ if one of the following situations hold

a) If $r \in B_{k_1,s}(l)$ and $p \neq i$.

b) If $k_2 < \frac{j-1}{2}$ and (($r = l$ and $p \in B_{k_2,j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r > l$ and $p = \sigma^+_{k_2+1,j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r < l$ and $p = \sigma^-_{k_2+1,j}(i)$)).

c) If $k_2 = \frac{j-1}{2}$and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r > l$ and $p = \sigma^+_{k_2+1,j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $r < l$ and $p = \sigma^-_{k_2+1,j}(i)$)).

d) If $k_2 > \frac{j-1}{2}$ and (($r = l$ and $p \neq i$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p \in B_{\frac{2k_2-(j-1)}{2},j}(i)$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p = \sigma^+_{\frac{2k_2-(j-1)}{2}+1,j}(i)$ with $r > l$) or ($r \in \sigma_{\frac{s}{2},s}(l)$ and $p = \sigma^-_{\frac{2k_2-(j-1)}{2}+1,j}(i)$ with $r < l$)).

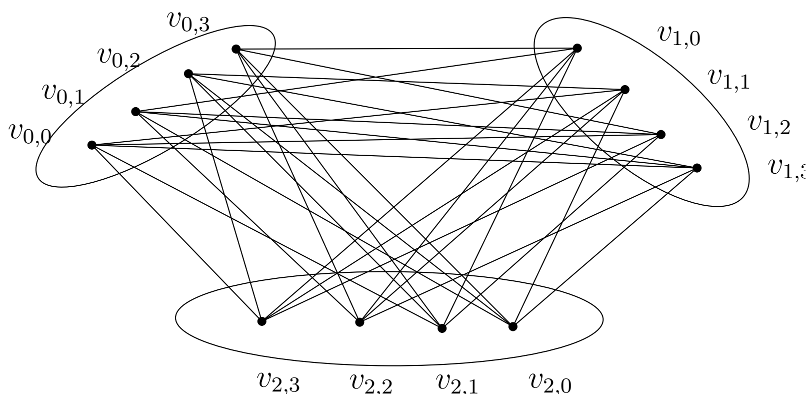

Figure 10: In the case(3) when $d = 5 = 2k_1(j-1) + 2k_2 + 1 = 2 \times 1 \times (3-1) + 2 \times 0 + 1$

It should be noted that the vertex sets of part (b), (c) and part (d) are disjoint. Therefore, $v_{i,l}$ will be either adjacent the vertices corresponding to part (a) and part (b) or (a) and part (c) or else (a) and part (d) according to whether $k_2 < \frac{j-1}{2}$, $k_2 = \frac{j-1}{2}$ or $k_2 > \frac{j-1}{2}$ respectively. In all these scenarios we get $d = 2k_1(j-1) + 2k_2 + 1$ as required. □

Lemma 2.3. $m_j(S_n, S_m) \leq \left\lceil \dfrac{n+m-3}{j-1} \right\rceil$ *where* $j, n, m \geq 3$.

Proof. Consider any red/blue coloring given by $K_{j \times s} = H_R \oplus H_B$, where $s = \left\lceil \dfrac{n+m-3}{j-1} \right\rceil$, such that H_R contains no red S_n. Let v be any vertex of $K_{j \times s}$. Then v is incident to at most $n-2$ red edge. Hence,

$$d_B(v) \geq \left\lceil \frac{n+m-3}{j-1} \right\rceil (j-1) - (n-2) \geq m-1$$

Therefore, H_B will contain a blue S_m. Hence the result. □

Lemma 2.4. $m_j(S_n, S_m) \geq \left\lceil \dfrac{n+m-4}{j-1} \right\rceil$ *where* $j, n, m \geq 3$.

Proof. Consider the red and blue coloring of $K_{j \times s}$ given by $K_{j \times s} = H_R \oplus H_B$, where $s = \left\lceil \dfrac{n+m-4}{j-1} \right\rceil - 1$, where all the vertices will have uniform red degree of $n-2$ or $n-3$ (this is possible by lemma 2.2). Then clearly H_B does not contain a red S_n. Let v be any vertex of $K_{j \times s}$. Then,

$$\begin{aligned} d_B(v) &= \left(\left\lceil \frac{n+m-4}{j-1} \right\rceil - 1 \right)(j-1) - (n-3) \\ &= \left\lceil \frac{n+m-4}{j-1} \right\rceil (j-1) - j + 1 - n + 3 \\ &\geq n + m - 4 - j - n + 4 \geq m - j \end{aligned}$$

Therefore, H_B will not contain a blue S_m. Hence the result. □

Lemma 2.5. $m_j(S_n, S_m) = \left\lceil \frac{n+m-4}{j-1} \right\rceil$ *if* $(n+m-4) \neq 0$ *mod* $(j-1)$ *where* $j, n, m \geq 3$.

Proof. We know that if $(n+m-4) \neq 0$ mod $(j-1)$ then $\left\lceil \frac{n+m-4}{j-1} \right\rceil = \left\lceil \frac{n+m-3}{j-1} \right\rceil$. Hence the result follows by lemma 2.3 and lemma 2.4.

□

Lemma 2.6. *Suppose that* $j, n, m \geq 3$. *Then,* $m_j(S_n, S_m) \leq \left\lceil \frac{n+m-4}{j-1} \right\rceil$ *provided that* $(n+m-4) = 0$ *mod* $(j-1)$ *with* j *is odd,* n *is odd and* $s = \frac{n+m-4}{j-1}$ *is odd.*

Proof. Consider any red/blue coloring given by $K_{j\times s} = H_R \oplus H_B$, where $s = \left\lceil \frac{n+m-4}{j-1} \right\rceil$, such that H_R contains no red S_n. Since, $j \times s \times (n-2)$ is odd, there will exist at least one vertex $v \in K_{j\times s}$ such it is not incident to $n-2$ red edges, as otherwise by handshake lemma $j \times s \times (n-2) = 2|E(H_R)|$, a contradiction. Hence,

$$d_B(v) \geq \left\lceil \frac{n+m-4}{j-1} \right\rceil (j-1) - (n-3) \geq m-1$$

Therefore, H_B will contain a blue S_m. Hence the result. □

Lemma 2.7. *Suppose that* $j, n, m \geq 3$. *Then,* $m_j(S_n, S_m) \geq \left\lceil \frac{n+m-3}{j-1} \right\rceil$ *provided that* $(n+m-4) = 0$ *mod* $(j-1)$ *with* j *is even or* $s = \frac{n+m-4}{j-1}$ *even or* n *is even.*

Proof. By lemma 2.3 and lemma 2.4, $K_{j\times s}$ where $s = \left\lceil \frac{n+m-3}{j-1} \right\rceil - 1 = \left\lceil \frac{n+m-4}{j-1} \right\rceil$, will have a $n-2$ regular subgraph on $K_{j\times s}$. Using this subgraph generate a red/blue coloring given by $K_{j\times s} = H_R \oplus H_B$, where all the edges of this subgraph are colored red and all other edges colored blue. Then clearly H_R is S_n - free. Furthermore, for any vertex $v \in K_{j\times s}$, $d_B(v) = \left(\frac{n+m-4}{j-1} \right)(j-1) - (n-2) = m-2$. Therefore, H_B will not contain a blue S_m. Hence the result. □

Theorem 2.1. *If $j \geq 3$ and $n, m \geq 2$ then,*

$$m_j(S_n, S_m) = \begin{cases} \left\lceil \dfrac{\max\{n,m\}-1}{j-1} \right\rceil & \text{if } n = 2 \text{ or } m = 2 \\ \left\lceil \dfrac{n+m-4}{j-1} \right\rceil & \text{if } n+m-4 = (j-1)s; j, s, n \\ & \text{are odd and } n, m \geq 3 \\ \left\lceil \dfrac{n+m-3}{j-1} \right\rceil & \text{otherwise} \end{cases}$$

Proof. The theorem clearly follows from lemmas 2.5, 2.6 and 2.7 as $m_j(S_2, S_m) = \left\lceil \dfrac{m-1}{j-1} \right\rceil$ (see Syafrizal et al 2005). □

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References

[1] A.P. Burger and J.H. van Vuuren, Ramsey numbers in Complete Balanced Multipartite Graphs. Part II: Size Numbers. *Discrete Mathematics*, **283** (2004), 45–49.

[2] P. Erdös, R.J. Faudree and R.H. Schelp, The size Ramsey numbers, *Period Mathematics Hungary*, **19** (1978), 161–173.

[3] R.J. Faudree and R.H. Schelp, Path-path Ramsey-type numbers for the complete bipartite graph, *Journal of Combinatorial Theory (B).*, **19** (1975), 161–173.

[4] C.J. Jayawardene ,E.T. Baskoro, L. Samarasekara and Syafrizal Sy, Size multipartite Ramsey numbers for stripes versus small cycles, *Electronic Journal of Graph Theory and Applications* **4(2)** (2016), 157–170.

[5] C.J. Jayawardene and L. Samarasekara, Size multipartite Ramsey numbers for stripes for $K_4 - e$ versus all graphs G up to 4 vertices, *Annals of Pure and Applied Mathematics* **13(1)** (2017), 9–26.

[6] A. Lusiani A., E.T. Baskoro, Syafrizal Sy and C.J. Jayawardene, Size multipartite Ramsey numbers for stars versus cycles, *Procedia Computer Science* **74** (2015), 27–31.

[7] S.P. Radziszowski, Small Ramsey numbers, *Electronic Journal of Combinatorics*, (rev **14**) (2014), DS1.

[8] Syafrizal Sy, E.T. Baskoro and S. Uttunggadewa, The size multipartite Ramsey number for paths, *Journal Combinatorial Mathematics and Combinatorial Computing* **55** (2005), 103–107.

[9] Syafrizal Sy, E.T. Baskoro, S. Uttunggadewa and H. Assiyatun, Path-path size multipartite Ramsey numbers, *Journal Combinatorial Mathematics and Combinatorial Computing* **71** (2009), 265–271.